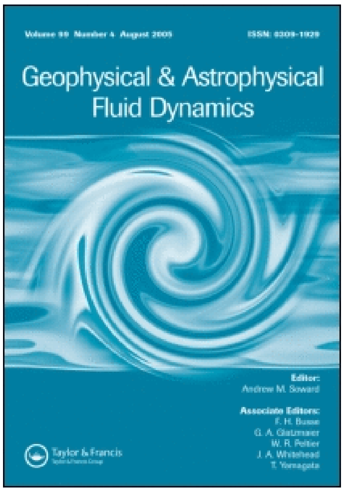

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MODAL GROWTH AND COUPLING IN THREE-DIMENSIONAL SPHERICAL CONVECTION

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The growth of spherical harmonic modes with increasing Rayleigh number *Ra* (a nondimensional measure of convective vigor) in numerical simulations of three-dimensional, Boussinesq, infinite Prandtl number, basally heated, spherical-shell convection is analyzed. Two regular polyhedral convective patterns with tetrahedral and cubic symmetry are examined. Apart from the dominant spherical harmonic modes which define the polyhedral patterns, the most important modes (in terms of modifying the convection as *Ra* increases) are ones whose wavenumbers (i.e., spherical harmonic degree and order) are exactly triple those of the dominant modes. Modes with wavenumbers that are five times those of the dominant modes also maintain large growth rates with increasing *Ra*. These results indicate the possibility that the spherical harmonic modes which are primarily responsible for modifying convection (e.g., narrowing the boundary layers) with increasing Rayleigh number, occur at wavenumbers that are odd integer multiples of those of the dominant modes. This suggests that an extended mean field method—wherein solutions in which only these modes are kept—may reasonably represent steady convection with regular polyhedral patterns up to relatively high *Ra*; such a method would entail a significant simplification in the analysis of nonlinear convection.

KEY WORDS: Mantle convection, convective patterns, three-dimensional, spherical geometry, mean field methods.

1. INTRODUCTION

Three-dimensional thermal convection is not only important for understanding the dynamics of planetary interiors, it is also a paradigm of self-organizing structures and patterns in nonlinear systems. Basally heated convection, in both plane layers and spherical shells, is known for establishing horizontal patterns with regular

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polygonal (in plane layers) or polyhedral (spherical shells) symmetries (e.g., Busse, 1975, 1978; Busse and Riahi, 1982; Machetel *et al.*, 1986; Houseman, 1988; Bercovici *et al.*, 1989a; Travis *et al.*, 1990). For spherical shells, Busse (1975) predicted that the regular polyhedral patterns predicted by slightly supercritical perturbation theory would persist to Rayleigh numbers in the strongly nonlinear regime. This prediction was verified numerically (Machetel *et al.*, 1986; Bercovici *et al.* 1989a) for several polyhedral patterns. However, while the basic polyhedral patterns persist, the structure of vertical and horizontal boundary layers changes significantly (i.e., the boundary layers narrow with increasing Rayleigh number). Thus, in addition to the spherical harmonic modes that define the polyhedral patterns, higher wavenumber modes become increasingly important as Rayleigh number increases. However, the manner in which these high wavenumber modes grow while preserving the structure of the polyhedral patterns is not clear. If the growth of high wavenumber modes is systematic, then it is possible that the way in which the convection planform changes with increasing convective vigor is predictable (at least for symmetric patterns).

In this study, we examine the growth of spherical harmonic modes for numerical simulations of convection. The numerical modelling is of three-dimensional, infinite Prandtl number (i.e., highly viscous), Boussinesq convection in a basally heated spherical shell. The thickness of the shell is typical of the mantles of the Earth, Mars and Venus (i.e., the inner to outer radius ratio $r_{bot}/r_{top} \approx 0.55$; Stevenson *et al.*, 1983). The numerical solutions are steady, maintain regular polyhedral symmetry, and are for Rayleigh numbers up to $Ra \approx 100 Ra_{cr}$ (where Ra_{cr} is the critical Ra for the onset of convection). We find that modes whose wavenumbers (i.e., spherical harmonic degree and order) are odd integer multiples of those of the dominant modes (where the dominant modes determine the convective pattern) may be the most important modes for modifying convection as Rayleigh number increases. The predominance of these modes, therefore, indicates that modal growth is systematic. This suggests that an extended mean field method—wherein solutions are only expanded in terms of these modes—could possibly model three-dimensional convection to high Rayleigh numbers with far fewer complications than a rigorous numerical model.

2. NUMERICAL METHOD AND SOLUTIONS

The numerical method employs a spectral-transform, Chebyshev collocation scheme to solve the three-dimensional equations of mass, momentum and energy conservation for a Boussinesq, infinite Prandtl number Pr fluid with constant viscosity and thermal conductivity. The approach is discussed in further detail by Glatzmaier (1984, 1988) and Bercovici *et al.* (1989a). Solutions are generated up to $Ra \approx 100 Ra_{cr}$ for two regular polyhedral convective patterns: one pattern has cubic symmetry (the pattern actually forms an octahedron) and the other has tetrahedral symmetry. All solutions are found to be steady at these Rayleigh numbers. The regular polyhedral solutions were predicted by perturbation theory (Busse, 1975; Busse and Riahi, 1982) and found numerically by Machetel *et al.* (1986) and Bercovici *et al.* (1989a). The solutions discussed in this paper are examined in further detail in Bercovici (1989) and Bercovici

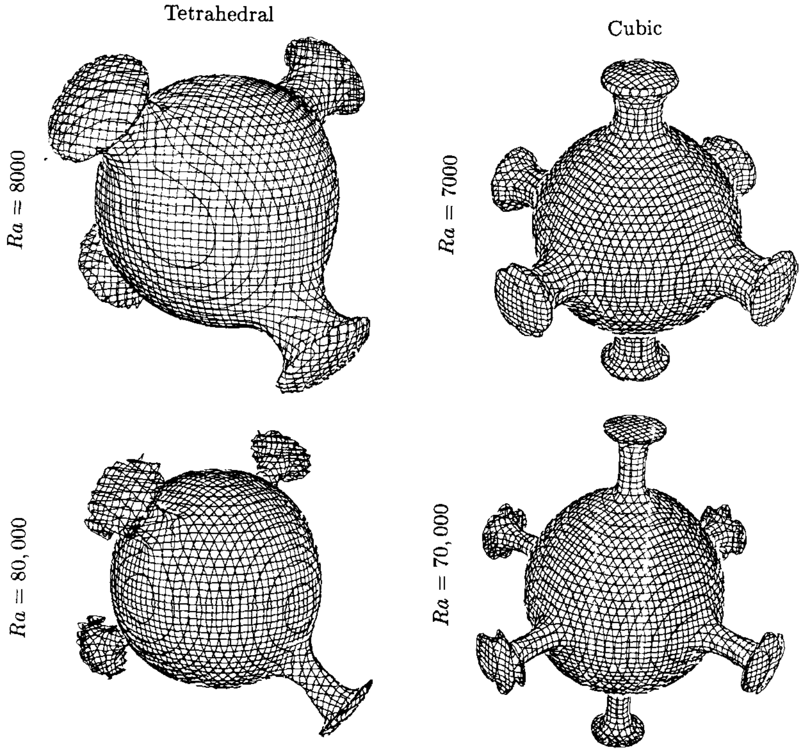

Figure 1 Three-dimensional isothermal surfaces for the tetrahedral and cubic patterns at two different Rayleigh numbers each. Protrusions represent columnar updwellings; surrounding depressions represent downwellings. The isotherms are with respect to nondimensional temperature Θ with $0 \leqslant \Theta \leqslant 1$.

et al. (1989a, 1991); these papers should be referred to for discussion of numerical resolution, convergence, etc.

The three-dimensional isothermal surfaces shown in Figure 1 illustrate the planforms of these solutions as well as the influence on boundary layer structure (both horizontal and vertical) of increasing Rayleigh number. As Raleigh number increases, the vertical boundary layers become narrower, indicating the growth of large wavenumber modes; however, the regular polyhedral patterns persist.

3. GROWTH OF SPHERICAL HARMONIC MODES

The spectral energy of spherical harmonic modes is measured in terms of the temperature variance for each spherical harmonic degree l and order m. If dimensionless temperature Θ (where the imposed temperature drop across the shell is the temperature scale) is expressed in a spherical harmonic series

$$\Theta(r,\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\Theta_l^m(r)Y_l^m(\theta,\phi), \tag{1}$$

(where r is radius, θ is colatitude, ϕ is longitude, Θ_l^m are complex coefficients, and Y_l^m are appropriately normalized spherical harmonic functions), then the temperature variance is

$$\langle\Theta^2\rangle_l^m=\int_{r_{\rm bot}}^{r_{\rm top}}\Theta_l^{m*}(r)\Theta_l^m(r)r^2\,dr. \tag{2}$$

It is also possible to measure spectral energy in terms of kinetic energy instead of temperature variance. The normalized kinetic energies and temperature variances are not necessarily equal for a given l and m; the small wavelength modes have generally smaller relative kinetic energies since the momentum field has broader and smoother features than the thermal field when Pr is very large. However, when the solutions are steady, the largest kinetic energies coincide with the largest temperature variances in l, m space; i.e., the kinetic energy has the same dominant modes as the temperature variance. This is not necessarily true when the solutions are unsteady, since a large thermal anomaly may appear that has not yet gathered enough buoyancy to affect the momentum field.

For the tetrahedral pattern, the dominant mode [i.e., the largest $\langle\Theta^2\rangle_l^m$ apart from $\langle\Theta^2\rangle_0^0$] occurs at $l=3$, $m=2$. The cubic pattern has two dominant modes, at $l=4$, $m=0$ and $l=4$, $m=4$. The ratio of the two modes $\langle\Theta^2\rangle_4^4/\langle\Theta^2\rangle_4^0$—as predicted by perturbation theory (Busse, 1975)—is always equal to 5/7 for all the Rayleigh numbers studied (Bercovici *et al.*, 1989a).

Figure 2a and Table 1 show temperature variance versus Rayleigh number for the second through ninth largest spherical harmonic modes of the tetrahedral pattern; the variances are normalized by the variance of the largest (tetrahedral) mode that occurs at $l=3$, $m=2$. The variances of all modes increase monotonically with Rayleigh number. The $l=4$, $m=0$ and $l=4$, $m=4$ modes are always approximately 10% of the tetrahedral modes and represent the small, indiscernible signature of a pattern with cubic symmetry; the ratio between the variances of these two modes is always close to 5/7, typical of the cubic pattern. The combination of the tetrahedral and cubic signatures characterizes a mixed mode solution which was predicted by Busse and Riahi (1988) and found numerically by Bercovici *et al.* (1989a).

The mode that undergoes the most dramatic growth with Ra is at $l=9$, $m=6$. This small wavelength mode reflects the narrowing of the upwelling and downwelling currents with increasing Rayleigh number. The variance for this mode approximates a power law dependence on Rayleigh number similar to the Nusselt number—Rayleigh number relationship (where the Nusselt number Nu is the ratio of total heat flow to

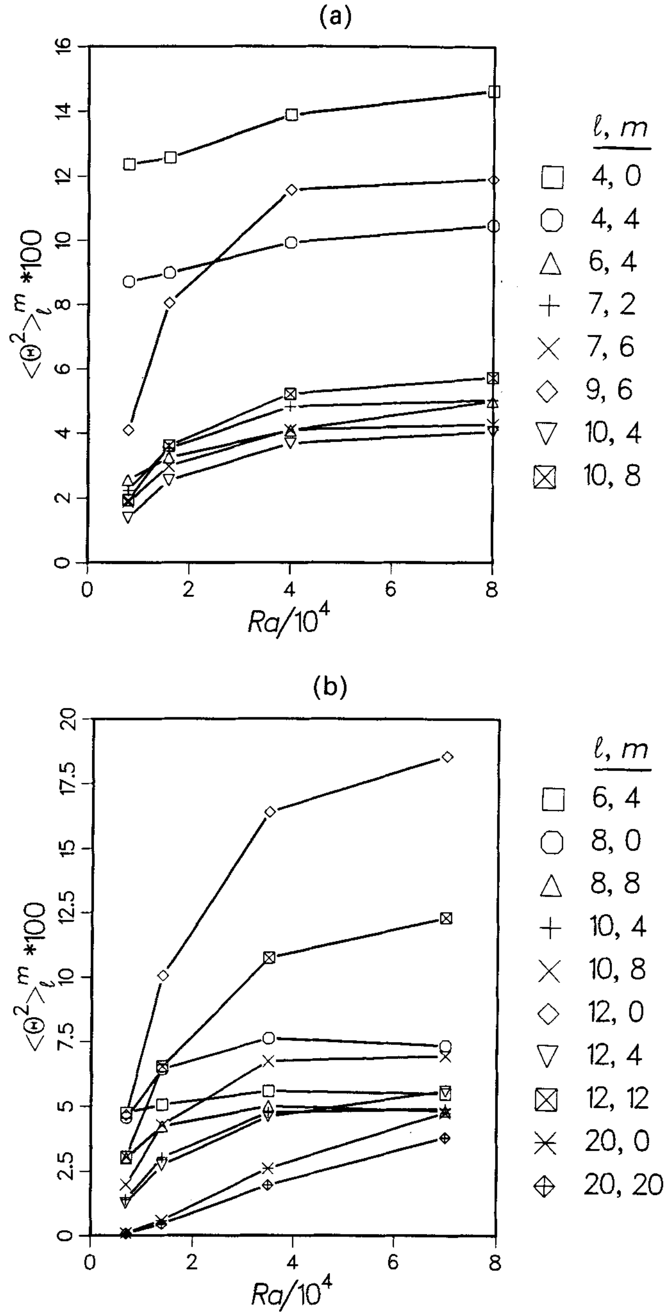

Figure 2 The largest nondominant temperature variances versus Rayleigh number for the a) tetrahedral and b) cubic patterns. The variances are normalized by the maximum variances (i.e., $\langle\Theta\rangle_3^2$ and $\langle\Theta\rangle_4^0$ in the tetrahedral and cubic cases, respectively) which are not shown on either figure; in addition, the $l=4, m=4$ variance is not shown for the cubic solutions since it has a constant normalized value of 5/7.

Table 1 The largest nondominant temperature variances $\langle\Theta^2\rangle_l^m$ for different Rayleigh numbers Ra for the tetrahedral and cubic patterns. Values correspond to those plotted in Figure 2; see caption to Figure 2 for discussion

		Tetrahedral			
l	m	$Ra=8000$	16000	40000	80000
4	0	12.360	12.560	13.880	14.630
4	4	8.709	8.978	9.925	10.460
6	4	2.547	3.238	4.076	4.988
7	2	2.233	3.522	4.819	5.025
7	6	1.867	2.981	4.093	4.269
9	6	4.092	8.041	11.560	11.900
10	4	1.387	2.539	3.678	4.040
10	8	1.924	3.599	5.219	5.728

		Cubic			
l	m	$Ra=7000$	14000	35000	70000
6	4	4.746	5.053	5.579	5.474
8	0	4.553	6.416	7.615	7.328
8	8	2.990	4.212	5.000	4.811
10	4	1.397	3.009	4.755	4.893
10	8	1.979	4.263	6.736	6.932
12	0	4.670	10.050	16.400	18.550
12	4	1.261	2.720	4.597	5.581
12	12	3.032	6.528	10.740	12.290
20	0	0.091	0.586	2.597	4.743
20	20	0.074	0.444	1.957	3.779

purely conductive heat flow)

$$\langle\Theta^2\rangle_9^6/\langle\Theta^2\rangle_3^2 \sim Ra^f. \tag{3}$$

For the higher Rayleigh numbers, f is 0.25, similar to the value of 0.26 for the power law exponent in the $Nu-Ra$ relationship (Bercovici *et al.*, 1989a). This implies that the growth of the $l=9$, $m=6$ mode with Rayleigh number is primarily responsible for the narrowing of the vertical boundary layers and the enhancement of heat transport.

The $l=9$, $m=6$ mode has the largest possible wavenumber of any of the products of a triple nonlinear interaction of the tetrahedral mode with itself. However, the conservation equations contain only quadratic nonlinearities, so a direct triple interaction is not possible. The $l=9$, $m=6$ mode must arise from an indirect triple product, namely the nonlinear interaction of the $l=3$, $m=2$ tetrahedral mode with the $l=6$, $m=4$ mode. The $l=6$, $m=4$ mode is the largest wavenumber product of the quadratic interaction of the tetrahedral mode with itself, and when this mode is dominant (as in some compressible convection solutions) the convective pattern has fourteen upwelling plumes and the solutions are metastable (i.e. they are stable for long yet limited periods of time; Bercovici *et al.*, 1991). Finally, it is also noteworthy that the $l=15$, $m=10$ mode has one of the largest growth rates with increasing Ra; its temperature variance increases by a factor of 28 between $Ra=8000$ and $Ra=80{,}000$.

Figure 2b and Table 1 show the third through twelfth largest temperature variances for the cubic pattern as functions of Ra. The variances are normalized by $\langle\Theta^2\rangle_4^0$. Since the two largest variances $\langle\Theta^2\rangle_4^4$ and $\langle\Theta^2\rangle_4^0$ have a constant ratio of 5/7 for all Rayleigh numbers, they are not shown in either Figure 2b or Table 1. The modes that grow fastest with increasing Ra, and eventually dominate the other modes shown, are the $l=12$, $m=0$ and $l=12$, $m=12$ modes. These are triple the wavenumbers of the two basic cubic modes, implying again that the cascade of spectral energy to higher wavenumbers is most concentrated through the indirect triple nonlinear interaction of the modes with themselves. The Rayleigh number dependence of these two modes analogous to (3) has exponents $f=0.18$ and 0.19 for the $l=12$, $m=0$ and $l=12$, $m=12$ modes, respectively. These are significantly less than 0.28 for the exponent of the $Nu-Ra$ relationship for the cubic pattern (Bercovici *et al.*, 1989a), thus the connection between the growth of these modes and enhancement of heat transport is not obvious. The products of interactions of the two cubic (i.e. $l=4$, $m=0$ and $l=4$, $m=4$) modes with each other do not grow at nearly the same rate, or to the same size as $l=12$, $m=0$ and $l=12$, $m=12$ modes; e.g., while the $l=12$, $m=4$ mode is among the ten largest modes shown in Figure 2b, it has only a moderate amount of energy, and the $l=12$, $m=8$ mode is not among the ten largest modes shown. Although the products of double interactions ($l=8$, $m=0$ and $l=8$, $m=8$) are some of the largest modes, they actually decrease slightly in energy from $Ra=35{,}000$ to $Ra=70{,}000$. The two smallest modes shown, at $l=20$, $m=0$ and $l=20$, $m=20$ have two of the largest growth rates with increasing Ra; both modes increase by a factor of 50 from $Ra=7000$ to 70,000. These modes represent the quintuple indirect interaction of the two dominant modes with themselves. Finally, it is of some interest that several modes of like l have identical percent growth between $Ra=7000$ and 70,000; e.g., modes with $l=8$ and $m=0$, 4 and 8 all grow 61% and modes with $l=10$ and $m=0$, 4 and 8 grow 250%.

4. DISCUSSION: SYMMETRY CONSIDERATIONS

That the triple product of the dominant modes of the tetrahedral and cubic patterns increase most rapidly with Rayleigh number and the quintuple products have very high growth rates with Ra, imply that odd nonlinear products of the dominant modes may be the most important modes for modifying convection (e.g., narrowing boundary layers, enhancing heat transport) as Ra increases. A partial explanation for this is that the modes which grow to modify and narrow the vertical currents must maintain the same symmetry as the dominant pattern. Only odd products of the dominant mode can have its symmetry since even products are symmetric where the dominant pattern is asymmetric and thus destructively interfere with the dominant pattern. For example, the mode which modifies the upwelling and downwelling regions of the tetrahedral ($l=3, m=2$) pattern must have odd symmetry about the equator, as does the tetrahedral pattern itself; since the even products of this mode with itself are of even spherical harmonic degree and order (and hence are equatorially symmetric), only the odd products can maintain the equatorial asymmetry of the dominant pattern.

Symmetry arguments have been previously used in analytical and numerical studies of spherical shell convection to reduce the number of spectral coefficients of a dependent variable. For example, assuming symmetry about the meridional plane $\phi=0$ insures that only the real components of Θ_l^m in (1) will be nonzero (e.g. Busse, 1975); or, in axisymmetric convection the assumption of equatorial symmetry limits nonzero modes to those with even l (e.g. Zebib *et al.*, 1980; 1985). However, the above symmetry argument is necessary but not sufficient for explaining the predominance of modes whose wavenumbers are odd integer multiples of those of the dominant modes. For example, modes that have the same symmetry planes as the tetrahedral mode are only required to have m be even, $m/2$ be odd and $l-m$ be odd. This can be seen by taking the spherical harmonic transform of a spherical step function that has tetrahedral planes of symmetry

$$F(\theta,\phi)=\begin{cases} 1 & \text{for } 0\leqslant\theta\leqslant\pi/2\text{:}\quad 0\leqslant\phi\leqslant\pi/2 \quad\text{and}\quad \pi\leqslant\phi\leqslant 3\pi/2, \\ & \text{for } \pi/2\leqslant\theta\leqslant\pi\text{:}\quad \pi/2\leqslant\phi\leqslant\pi \quad\text{and}\quad 3\pi/2\leqslant\phi\leqslant 2\pi; \\ -1 & \text{elsewhere} \end{cases} \tag{4}$$

to yield

$$F_l^m \sim \left[1+(-1)^m-2\cos\left(\frac{m\pi}{2}\right)\right]\left\{\int_0^1 P_l^m(\cos\theta)d(\cos\theta)-\int_{-1}^0 P_l^m(\cos\theta)d(\cos\theta)\right\}, \tag{5}$$

where the P_l^m are the associated Legendre functions. The term in square brackets is only nonzero for m even and $m/2$ odd; the term in curly brackets is only nonzero for those P_l^m which are odd functions of $\cos\theta$, i.e., P_l^m with $l-m$ odd. There are many modes that satisfy these tetrahedral symmetry requirements (e.g. $l=7$, $m=6$), thus symmetry arguments alone do not explain the predominance of the "odd product" modes. The apparent predominance of these modes must, therefore, be due to another physical mechanism.

5. CONCLUSION: IMPLICATIONS FOR A MODIFIED MEAN FIELD METHOD

The numerical results presented in this study suggest that for convection with regular polyhedral patterns, the nondominant spherical harmonic modes responsible for modifying convection (i.e., narrowing boundary layers, enhancing heat transport, etc.) as Rayleigh number increases have wavenumbers that are odd integer multiples of the wavenumbers of the dominant modes. This implies that the growth of high wavenumber modes is indeed systematic and perhaps predictable. However, these hypotheses are in need of a rigorous mathematical proof beyond symmetry arguments. A recently developed approach for analyzing convective patterns using group theory (McKemzie, 1988) may offer the best method for providing this proof. Given such a proof, it may be possible to study three-dimensional convection in a very cost effective manner. Instead of assuming a horizontal convective pattern with only one wavenumber as in mean field theory (e.g. Roberts, 1966; Olson, 1981; Quareni and Yuen, 1988), or carrying hundreds or thousands of modes as in a full numerical analysis, one might only need to keep the terms of a spectral series whose wavenumbers are odd products of the wavenumbers of the dominant modes. For a tetrahedral pattern, one would only need to carry modes with spherical harmonic degree and order ($l=3$, $m=2$), ($l=9$, $m=6$), ($l=15$, $m=10$), etc.; for a truncated series with maximum degree 30, one would keep 5 complex spherical harmonic terms, as opposed to almost 500 complex spherical harmonics for a full numerical analysis. This method may present an effective way of computing high Rayleigh number three-dimensional solutions with orders of magnitude reduction in memory and time requirements as compared to a full numerical scheme.

This method would be, in effect, a multimode mean field method. However, it would be significantly different from previous multimode mean field theories. In the study of Toomre *et al.* (1982)—where two and three modes were maintained in a spectral representation—the modes were chosen by first assuming a fundamental mode which determines the basic convection pattern (e.g. the $l=3$, $m=2$ mode in the tetrahedral pattern) and then determining the second and third "overtone" modes, i.e., the next highest wavenumber modes that are nonzero according to the selection rules of the interaction coefficients. The interaction coefficients are essentially the nonlinear terms of the governing equations in spectral or wavenumber space and involve the integral of the triple products of the orthonormal functions; e.g., the interaction coefficients in spherical convection are proportional to

$$\int_{0}^{2\pi}\int_{-1}^{+1} Y_l^\lambda Y_m^\mu Y_n^{\nu *}\, d(\cos\theta)d\phi.$$

Because of symmetry or orthogonality, these coefficients are zero for many choices of wavenumbers. For example, the above integral is zero if $\lambda+\mu\neq\nu$ (because $Y_l^\lambda \sim e^{i\lambda\phi}$) or if $l+m<n$ (because $Y_l^\lambda Y_m^\mu$ can be expressed as a spherical harmonic series with maximum degree $l+m$ hence all terms in this series are orthogonal to Y_n^ν if $n>l+m$). These are a few examples of the selection rules; for a more complete discussion, see Ellsaesser (1966).

In this study we find that there are many relatively low wavenumber modes which obey the selection rules that are not as significant as higher wavenumber modes. For example, the first "overtone" mode to the tetrahedral mode involves the $l=6$, $m=4$ mode which 1) does not coincide with tetrahedral symmetry and 2) is less important than the higher wavenumber $l=9$, $m=6$ mode. Therefore, beyond symmetry and orthogonality, there are apparently other selection rules inherent in the physics as implied here. A multimode mean field method based on the results of this study would employ a more stringent set of selection rules and thus a smaller choice of overtone modes with which to construct a solution.

A few caveats about this proposed multimode mean field method are worth enumerating. Like a single mode mean field method, it does not represent rigorous convection. The method would assume that the polyhedral pattern is always stable, and thus would preclude bifurcations of steady solutions and possibly the onset of time dependence. Further, it would be unwise to apply this method to convection with internal heating which does not appear to have regular polyhedral patterns beyond modest Rayleigh numbers if at all (Bercovici *et al.*, 1989b, c, 1990; Schubert *et al.*, 1990; Glatzmaier *et al.*, 1990). Finally, although this method would expand dependent variables in odd product modes, it will still be necessary to account for intermediary even product modes to allow for the cascade of energy to high wavenumbers via nonlinear interactions. This requirement may dilute the usefulness of the modified mean field method in a purely Galerkin scheme, but a transform method would be unaffected since it accounts for nonlinear interactions in physical space.

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