Linear stability analysis of Richter rolls

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[1] The planform of sublithospheric convection is studied by a 3-D linear stability analysis of longitudinal rolls in the presence of vertical shear. Rayleigh numbers up to 10^6 are considered. The transition Rayleigh number over which longitudinal rolls are unstable is derived as a function of surface velocity. Effects of nonlinear vertical shear are shown to be insignificant unless nonlinearity becomes extreme. Our results are in good agreement with previous studies, except at high Rayleigh numbers where the meaning of roll stability becomes ambiguous owning to inherently time-dependent convection. When applied to small-scale convection in the upper mantle, our results suggest that the roll stability is very sensitive to plate velocity if the upper mantle viscosity is relatively high ($\sim 10^{20}$ Pa s); Richter rolls may be expected only beneath fast-moving plates such as the Pacific plate. *INDEX TERMS:* 3010 Marine Geology and Geophysics: Gravity; 3040 Marine Geology and Geophysics: Plate tectonics (8150, 8155, 8157, 8158); 8121 Tectonophysics: Dynamics, convection currents and mantle plumes; 8180 Tectonophysics: Tomography. **Citation:** Korenaga, J., and T. H. Jordan, Linear stability analysis of Richter rolls, *Geophys. Res. Lett.*, *30*(22), 2157, doi:10.1029/2003GL018337, 2003.

1. Introduction

[2] Recent high-resolution tomography of the Pacific upper mantle [e.g., *Katzman et al.*, 1998] has revived theoretical interest in the dynamics of sublithospheric convection [*Korenaga and Jordan*, 2002a, 2002b, 2003]. One important issue regarding sublithospheric convection is its three-dimensional convective pattern; a theoretical understanding of possible planforms is valuable with regard to the geophysical detection of small-scale convection. The planform of sublithospheric convection is believed to be strongly affected by the presence of vertical shear associated with plate motion [e.g., *Richter*, 1973]. To minimize the interference with the background flow, convection cells tend to align themselves in parallel with the direction of background flow. The dynamics of small-scale convection is completely decoupled from the large-scale flow in the perfect alignment of such longitudinal convection rolls [e.g., *Jeffreys*, 1928]. The critical Rayleigh number for marginal stability is the smallest for such planform [*Ingersoll*, 1966]. The planform of finite-amplitude convection is a more complicated matter; there is some critical intensity of shear, over which longitudinal rolls are stable [e.g., *Clever and Busse*, 1977]. Because of its relevance to atmospheric and oceanographic sciences as well as engineering applications, the physics of thermal convection with vertical shear has been studied by a number of scientists [e.g., *Hathaway and Somerville*, 1986; *Domaradzki and Metcalfe*, 1988; *Clever and Busse*, 1992]. Naturally enough, most of previous studies have focused on the dynamics of low Prandtl number fluid. The Prandtl number is the ratio of momentum diffusivity to thermal diffusivity; a low Prandtl number means that heat diffuses very quickly compared to velocity.

[3] The Prandtl number of the Earth's mantle is essentially infinite ($\sim 10^{24}$), and there are only a few studies on the planform of infinite Prandtl number convection with vertical shear. An early numerical investigation was presented by *Richter* [1973] for low Rayleigh numbers, which was later extended to higher Rayleigh numbers by laboratory experiments [*Richter and Parsons*, 1975]. Longitudinal rolls in mantle convection are thus often referred as "Richter rolls". Their main objective was, however, to characterize the response time of convective planform to a change in surface velocity, and less attention was paid to the stability condition for longitudinal rolls. *Cserepes and Christensen* [1990] studied the stability of Richter rolls using 3-D numerical calculations, but the parameter space they explored is limited.

[4] Thus, we do not have yet a scaling law for the planform of small-scale convection in the Earth's mantle. Furthermore, the type of vertical shear considered in *Richter and Parsons* [1975] and *Cserepes and Christensen* [1990] corresponds to a complete return flow in the upper mantle, and we want to know the stability condition in more general circumstances. In order to derive such a general scaling law, we conduct a 3-D linear stability analysis of 2-D steady-state convection with Rayleigh numbers up to 10^6. The relevance of this stability analysis to the planform of convection is the following. At the onset of convection (i.e., marginally stable state), we already know that in the presence of vertical shear, the planform should be dominated by longitudinal rolls [*Ingersoll*, 1966]. An important question is the stability of these longitudinal rolls with respect to various three-dimensional perturbations that are generated by the evolution of incipient convection to finite-amplitude convection. If our stability analysis indicates that longitudinal rolls are unstable to such perturbations for a given vertical shear, then, the planform of fully-evolved convection is expected to deviate from longitudinal rolls and most likely become multimodal [e.g., *Busse and Whitehead*, 1971]. The initiation of sublithospheric convection prevents

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further thickening of oceanic lithosphere, so the height of convection system is expected to be nearly constant, which justifies the use of 2-D convection rolls as a reference state in our stability analysis.

[5] Because strong temperature dependency of mantle rheology results in a nearly isoviscous convection beneath a rigid lid [e.g., *Solomatov*, 1995], we limit ourselves to constant viscosity in the following analysis. Weakly variable viscosity in sublithospheric convection probably has some additional effects on stability. Our results serve as the reference and the limiting case, which will be required to quantify the effect of variable viscosity.

2. Theoretical Formulation

[6] A fluid with infinite Prandtl number is bounded above and below by rigid horizontal boundaries, and temperature is fixed at T_s on the top and at T_0 on the bottom. Consider longitudinal convection cells aligned in parallel with one of the horizontal axes (hereinafter the y-axis). A unit aspect ratio is assumed for a convection cell. The aspect ratio of 2-D steady-state convection bounded by rigid surfaces is very close to unity near marginal stability [*Chandrasekhar*, 1981]. Because its cross section does not depend on y, its steady-state dynamics can be described generally by 2-D functions, such as $\mathbf{u}_0(x, z)$ for velocity and $\theta_0(x, z)$ for temperature. By denoting background vertical shear by $V(z)$, the complete 3-D velocity field can be expressed as

$$\mathbf{u}_0(x,z) = [u_0(x,z), V(z), w_0(x,z)]. \tag{1}$$

The 2-D vector field, (u_0, w_0), is generated by thermal buoyancy corresponding to θ_0, and we assume that V is generated by boundary conditions and/or a lateral pressure gradient. Length and time are normalized by a system depth, D, and a diffusion time, D^2/κ, respectively, where κ is thermal diffusivity. Velocity is thus normalized by κ/D. Temperature is normalized by $\Delta T(\equiv T_0 - T_s)$. Superscript * denotes normalized variables. The convection system is characterized by two non-dimensional parameters, the Rayleigh number and the Peclet number. The former is defined as

$$Ra = \frac{\alpha\rho_0 g\Delta T D^3}{\kappa\mu_0} \tag{2}$$

where α is the coefficient of thermal expansion, g is gravitational acceleration, ρ_0 is reference density at T_0, and μ_0 is fluid viscosity. The Peclet number is nondimensionalized velocity based on thermal diffusion. A general expression for vertical shear, with the surface velocity of V_0^* and the bottom velocity of 0, is the following Couette flow solution:

$$V^*(z) = V_0^*\left[(1-\chi)z^* + \chi(z^*)^2\right], \tag{3}$$

where $\chi = C/V_0^*$, and an external pressure gradient is included in C. A linear shear profile corresponds to $\chi = 0$. For a complete return flow, $\chi = 3$. We call $V_0^*(=V_0 D/\kappa)$ the surface Peclet number.

[7] The outline of the present linear stability analysis is similar to that of *Korenaga and Jordan* [2001, section 5.1]. Only key information is given here. Among the governing equations for infinite Prandtl number thermal convection, a temporal derivative appears only in the energy equation, and a stability analysis can be formulated by perturbing this energy balance [e.g., *Busse*, 1967; *Clever and Busse*, 1977; *Korenaga and Jordan*, 2001]. Assuming the temporal evolution of perturbation temperature as $\theta^* \propto \exp(\lambda t^*)$, and using a finite-dimensional approximation, one can construct an eigensystem with eigenvalues λ. The stability condition for the longitudinal rolls is given by $\max[\mathrm{Re}(\lambda)] < 0$, i.e., the growth exponent is negative.

[8] 2-D steady-state solutions, u_0^*, w_0^*, and θ_0^*, are obtained numerically by iterating between the Stokes flow equation and the steady-state energy equation (see Figure 5a of *Korenaga and Jordan* [2001] for examples of θ_0^*). Because a steady-state solution does not depend on the y-coordinate, a general 3-D perturbation in our model can be decomposed into the sum of single-mode perturbations, each of which has a fixed y-coordinate wavenumber, ψ. Such single-mode temperature perturbation may be expressed as

$$\theta^* = \sum_{k,l} \beta_{kl} \begin{Bmatrix} \cos(k\pi x^*) \\ \sin(k\pi x^*) \end{Bmatrix} \sin(l\pi z^*) \begin{Bmatrix} \cos(\psi y^*) \\ \sin(\psi y^*) \end{Bmatrix}, \tag{4}$$

where k and l are positive integers, except that $k = 0$ must be included for $\cos(k\pi x^*)$ to make a complete set of perturbation modes. Because a steady-state solution and the above general perturbation share the same kind of periodic symmetry, the eigensystem can be decoupled into two subsystems, the one with symmetric modes (i.e., with cos $(k\pi x^*)$ terms) and the other with antisymmetric modes (with sin $(k\pi x^*)$ terms). The largest eigenvalue for the antisymmetric system turns out to be zero, corresponding to standing perturbation [*Busse*, 1967], so we will consider the symmetric system only. No further simplification can be made except for some special cases. When $V^* = 0$ or $\psi = 0$, for example, the subsystems can be further decoupled into even $k + l$ modes and odd $k + l$ modes, due to the rotational symmetry of a steady-state solution. The $\cos(\psi y)$ modes and the sin (ψy) modes are also decoupled in the case of linear shear. A solution for a 3-D perturbation velocity corresponding to the above normal-mode temperature perturbation is derived by *Busse* [1967], and the eigensystem can be constructed analytically once the steady-state temperature is expressed in terms of Fourier coefficients.

[9] The symmetric-mode eigensystem is solved using all modes with $k + l < N$, starting with $N = 15$ and increasing N until convergence is achieved. Our criterion for convergence is, if $|\mathrm{Re}(\lambda_N)| < 1$, $|\mathrm{Re}(\lambda_{N-2} - \lambda_N)| < 0.01$, and otherwise, $|\mathrm{Re}(\lambda_{N-2} - \lambda_N)/\mathrm{Re}(\lambda_N)| < 0.01$.

3. Results

[10] For each 2-D steady-state solution, a growth exponent, $\max[\mathrm{Re}(\lambda)]$, was first calculated as a function of wavenumber ψ. The 3-D stability of the given 2-D solution was then determined by taking the maximum of this wavenumber-growth exponent curve. Because of the possibility of multiple local maxima, we made a systematic grid search for $\psi = 0$–20 with $\Delta\psi = 0.5$ to identify an interval containing the global maximum, and used Brent's method [*Press et al.*, 1992] to locate λ_{max} with the tolerance of 0.01 in ψ. This procedure was repeated for $Ra = 10^{3.6}$–10^6 and $V_0^* = 0$–10^3. For most cases convergence is attained with $N < 30$. Results for linear shear are shown in Figure 1. Note

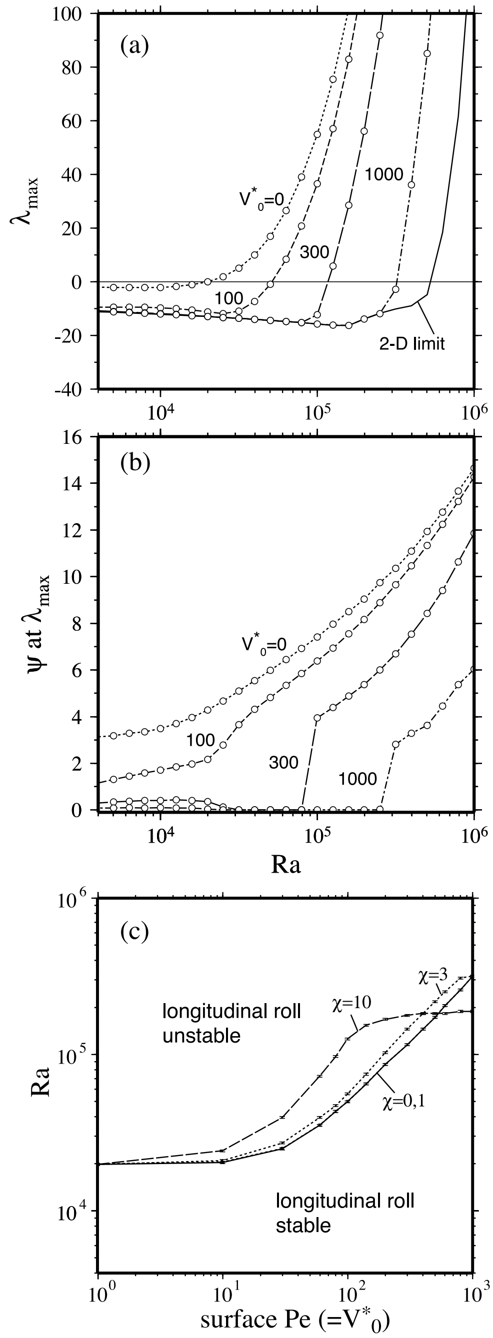

Figure 1. 3-D stability of 2-D steady-state convection in the presence of linear shear (χ = 0). The maximum growth exponent and the corresponding out-of-plate wavenumber are shown in (a) and (b), respectively. Note that the stability curve for purely 2-D perturbations (solid line in (a)) limits the 3-D stability of longitudinal rolls. An abrupt change in ψ at λ_{max} seen for large V_0^* is caused by moving from a maximum at ψ = 0 (i.e., 2-D limit) to another maximum at ψ > 0. (c) The transition Rayleigh number for stable to unstable longitudinal rolls is plotted as a function of the surface Peclet number.

that 3-D perturbations are always more unstable than purely 2-D perturbations. An abrupt change in ψ at λ_{max} seen for large V_0^* (Figure 1b) is caused by moving from a maximum at ψ = 0 (i.e., 2-D limit) to another maximum at ψ > 0. The 2-D limit indicates that 2-D convection becomes inherently time-dependent for $Ra > \sim 5 \times 10^5$ [e.g., *Korenaga and Jordan*, 2001]. Vertical shear cannot affect this time dependency because the influence of shear vanishes in the eigensystem at ψ = 0. For each V_0^* there exists a transition Rayleigh number over which λ_{max} becomes positive. The stability regime of longitudinal rolls is bounded by this transition Rayleigh number (Figure 1c).

[11] Results for nonlinear shear profiles with χ = 1, 3, and 10 are also shown in Figure 1c. The stability diagram for χ = 1 is almost identical with the linear case, and the difference between χ = 1 and 3 is also small. The effect of nonlinearity becomes significant only with very large χ, which does not appear to be realistic for mantle flow beneath oceanic plates.

4. Discussion and Conclusion

[12] One way to verify our stability analysis of longitudinal rolls is to test its consistency with previous studies, though the number of appropriate data to be compared is very small as noted earlier. *Richter and Parsons* [1975] reported 31 pairs of the Rayleigh number and the Peclet number, for which stable longitudinal rolls were observed. They did not report unstable pairs. Because their experiments were designed to study the transient behavior of longitudinal rolls, it would have been difficult, with their experiment setup, to distinguish between a truly unstable condition and a stable condition with a very long transient timescale. *Cserepes and Christensen* [1990] reported one stable pair and one unstable pair. For $Ra < 3 \times 10^5$, these data can be explained fairly well by our stability diagram (Figure 2). A few minor discrepancies observed within this

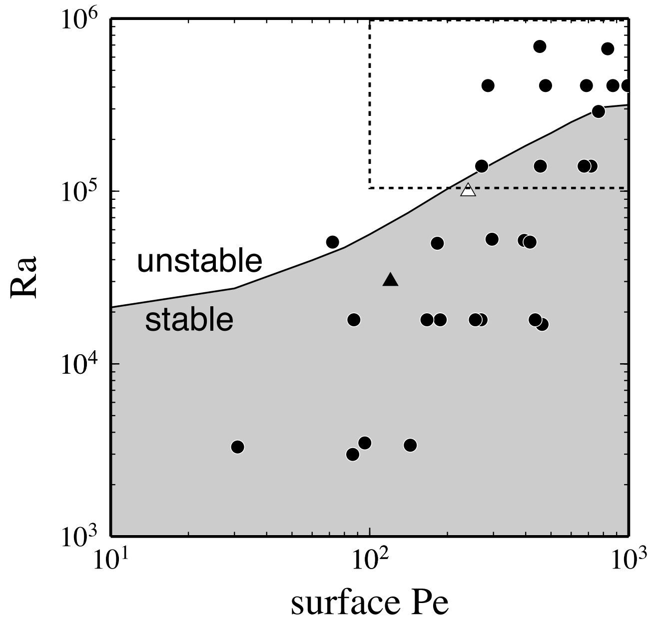

Figure 2. Comparison with previous work. The stability regime for χ = 3 is shaded. Solid circles are stable pairs of *Ra* and *Pe* from *Richter and Parsons* [1975]. Solid and open triangles are stable and unstable pairs, respectively, from *Cserepes and Christensen* [1990]. Dashed box indicates the parameter space relevant to small-scale convection in the upper mantle. See text for discussion.

range of Ra is probably owing to (1) experimental error in *Ra* (∼5%) and *Pe* (∼10%) [*Richter and Parsons*, 1975], (2) the use of a free-slip bottom and a small computational domain in the work of *Cserepes and Christensen* [1990], and (3) the use of a unit aspect ratio in our stability analysis whereas slightly smaller aspect ratios are expected for convection with high *Ra*. Note that the unstable pair of *Cserepes and Christensen* [1990] (open triangle) is also inconsistent with the nearby stable pairs of *Richter and Parsons* [1975] (Figure 2).

[13] On the other hand, stable rolls observed at $Ra > 4 \times 10^5$ with surface *Pe* lower than 10^3 clearly indicate the inadequacy of our linear stability analysis at high Rayleigh numbers. However, we also note that the definition of 'stable rolls' becomes vague at high Rayleigh numbers. For $Ra > \sim 5 \times 10^5$, a steady-state solution no longer exists (Figure 1a), so time-independent longitudinal rolls cannot exist. Indeed, what *Richter and Parsons* [1975] observed at $Ra \geq 1.4 \times 10^5$ is 'approximate' longitudinal rolls, in which each roll evolves with time (see Figure 9 of *Richter and Parsons* [1975]).

[14] Temperature variations associated with sublithospheric small-scale convection are of the order of 100 K, and the viscosity of sublithospheric mantle is probably in the range of 10^{19}–10^{20} Pa s [*King*, 1995; *Korenaga and Jordan*, 2002b]. With $\alpha = 3 \times 10^{-5}$ K^{-1}, $\rho_0 = 3300$ kg m^{-3}, $g = 9.8$ m s^{-2}, $\kappa = 10^{-6}$ m^2 s, and $D = 560$ km (assuming the steady-state lithospheric thickness of 100 km), the Rayleigh number for small-scale convection in the upper mantle is of the order of 10^5–10^6. These upper-mantle values also indicate that the Peclet numbers of 10^2 and 10^3 are equivalent to the plate velocities of ∼0.6 cm yr^{-1} and ∼6 cm yr^{-1}, respectively. Our stability diagram suggests that, if the Rayleigh number for small-scale convection is less than 4×10^5, whether Richter rolls form or not is very sensitive to plate velocity. For $Ra = 10^5$, for example, stable rolls form if plate velocity is greater than ∼1 cm yr^{-1}. For $Ra = 3 \times 10^5$, however, the minimum plate velocity for stable rolls becomes as high as ∼5 cm yr^{-1}. In this case, Richter rolls may be expected only beneath fast-moving plates such as the Pacific plate. For higher Rayleigh numbers, the definition of roll stability becomes nebulous as discussed above. On the basis of the work of *Richter and Parsons* [1975], the sensitivity to plate velocity appears to decrease considerably for $Ra > 4 \times 10^5$; for this range of the Rayleigh number, (time-dependent) Richter rolls may form even if plate velocity is only 1 cm yr^{-1}.

[15] By focusing on a simple situation (i.e., isoviscous and unit aspect ratio), we are able to conduct a comprehensive 3-D stability analysis. Considering the extent of parameter space we need to explore, we do not expect that this approach works similarly well for more realistic variable viscosity cases, for which numerical simulation may be preferable. Our results will be useful as a solid benchmark when conducting a scaling analysis with such simulation studies.

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